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On the characters of Wenzl's $(3, l)$-diagram representations of the Iwahori-Hecke algebras at $\sqrt[l]{1}$

Hiroyuki Yamane
Seiji Kono

§ 1. Wenzl's (k, l)-diagram representations

(1.1) *Throughout this note, we assume q to be a non-zero complex number.* Let $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, $\mathbf{C}$ be the set of natural numbers, the additive group of integers, the field of real numbers, the field of complex numbers, respectively, For $n \in \mathbf{N}$, define $\mathbf{H}_n(q)$ to be the $\mathbf{C}$-algebra (with 1) by the generators T_i, $1 \leq i \leq n-1$, and the relations:

$$(T_i - q)(T_i + 1) = 0,\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1},\ T_iT_j = T_jT_i\ (|i-j| \geq 2).$$

The algebra $\mathbf{H}_n(q)$ is called the *Iwahori-Hecke algebra* af type A_n. Let $\mathbf{C}^{\times} = \mathbf{C} \setminus \{0\}$ and $\mathbf{N}' = \mathbf{N} \setminus \{1\}$. Let $\mathbf{l} : \mathbf{C}^{\times} \to \mathbf{N}' \cup \{\infty\}$ be the map such that

$$\mathbf{l}(q) = \begin{cases} \min\{a \in \mathbf{N}' \mid q^a = 1\} & \text{if } \prod_{r=2}^{s}(\sum_{t=0}^{r-1} q^t) = 0 \text{ for some } s \in \mathbf{N}', \\ +\infty & \text{otherwise.} \end{cases}$$

Let $\mathbf{Z}_+ = \{0\} \cup \mathbf{N}$. Put $\mathbf{Z}_+^{\infty} = \{(x_1, x_2, \ldots) \in \mathbf{Z}^{\infty} \mid x_i \in \mathbf{Z}_+ (i \in \mathbf{N})\}$. For $i \in \mathbf{N}$, let $p_i : \mathbf{Z}_+^{\infty} \to \mathbf{Z}_+$ be the map such that $p_i(x_1 x_2, \ldots) = x_i$. An element λ of $\mathbf{Z}_+^{\infty}$ is called a *partition* if $p_i(\lambda) \geq p_{i+1}(\lambda)$ for any $i \in \mathbf{N}$ and there exists $j \in \mathbf{N}$ such that $p_j(\lambda) = 0$. Let Λ be the set of partitions. Let $\mathbf{k} : \Lambda \to \mathbf{Z}_+$ be the map such that $\mathbf{k}(\lambda) = \min\{j \in \mathbf{N} \mid p_j(\lambda) = 0\} - 1$. Let $\mathbf{n} : \Lambda \to \mathbf{Z}_+$ be the map such that $\mathbf{n}(\lambda) = \sum_{i=1}^{+\infty} p_i(\lambda)$. For $n \in \mathbf{Z}_+$, let $\Lambda_n = \{\lambda \in \Lambda | \mathbf{n}(\lambda) = n\}$. For $k \in \mathbf{Z}_+$, let $\Lambda^k = \{\lambda \in \Lambda | \mathbf{k}(\lambda) = k\}$.

(1.2) Let $\phi = (0, 0, \ldots) \in \Lambda$, and let $\Lambda' = \Lambda \setminus \{\phi\}$. For $l \in \mathbf{N}' \cup \{+\infty\}$, let

$$\Lambda^{(l)} = \{\phi\} \cup \{\lambda \in \Lambda' \mid \mathbf{k}(\lambda) \leq l-1,\ p_1(\lambda) - p_{\mathbf{k}(\lambda)}(\lambda) \leq l - \mathbf{k}(\lambda)\}.$$

For k, $1 \leq k \leq l-1$, let $\Lambda^{(k,l)} = \cup_{a=1}^{k}(\Lambda^a \cap \Lambda^{(l+a-k)})$. Note that $\Lambda^{(l)} = \cup_{1 \leq k \leq l-1} \Lambda^{(k,l)}$ and $\Lambda^{(+\infty)} = \Lambda$. Let $\Lambda_n^{(l)} = \Lambda^{(l)} \cap \Lambda_n$ and $\Lambda_n^{(k,l)} = \Lambda^{(k,l)} \cap \Lambda_n$. The element $\lambda \in \Lambda$ will also be denoted by

$$[1^{|\{i|p_1(\lambda)=1\}|} 2^{|\{i|p_1(\lambda)=2\}|} \ldots]$$

if $\lambda \in \Lambda'$, and [0] if $\lambda = \phi$. For example, we have $\Lambda_0^{(l)} = \{[0]\}$, $\Lambda_1^{(l)} = \{[1]\}$ and $\Lambda_2^{(l)} = \{[2], [1^2]\}$.

(1.3) For μ, $\lambda \in \Lambda$, we write $\mu \subset^{(k,l)} \lambda$ if μ, $\lambda \in \Lambda^{(k,l)}$ and $p_i(\mu) \leq p_i(\lambda)$ for any i, $1 \leq i \leq k$. We write $\mu \subset_{+1}^{(k,l)} \lambda$ if $\mu \subset^{(k,l)} \lambda$ and $\mathbf{n}(\mu) = \mathbf{n}(\lambda) - 1$. For $\mu \subset_{+1}^{(k,l)} \lambda$, put

$$\mathbf{r}(\mu, \lambda) = \sum_{i=0}^{+\infty} i \cdot (p_i(\lambda) - p_i(\mu)), \quad \text{and} \quad \mathbf{c}(\mu, \lambda) = p_{\mathbf{r}(\mu \subset_{+1}^{(k,l)} \lambda)}(\lambda).$$

(1.4) For $\mu \subset^{(k,l)} \lambda$, we set $\mathbf{n}(\lambda/\mu) = \mathbf{n}(\lambda) - \mathbf{n}(\mu)$, and call an element $(\lambda_0, \lambda_1, \ldots, \lambda_{\mathbf{n}(\lambda/\mu)})$ of $\Lambda_{\mathbf{m}(\lambda)} \times \Lambda_{\mathbf{m}(\lambda)+1} \times \cdots \times \Lambda_{\mathbf{n}(\lambda)}$ a (k,l)-*standard tabuleau* of λ/μ if $\lambda_0 = \mu$, $\lambda_{\mathbf{n}(\lambda/\mu)} = \lambda$ and $\lambda_i \subset_{+1}^{(k,l)} \lambda_{i+1}$, $1 \leq i \leq \mathbf{n}(\lambda/\mu) - 1$. Let $\mathrm{STab}^{(k,l)}(\lambda)$ denote the set of (k,l)-standard tableaux of λ/μ.

(1.5) A standard tabuleau $(\lambda_0, \lambda_1, \ldots, \lambda_{\mathbf{n}(\lambda/\mu)}) \in \mathrm{STab}^{(k,l)}(\lambda/\mu)$ will also be denoted by the table such that the i-th Arabic figure is put on $(\mathbf{r}(\lambda_{i-1}, \lambda_i), \mathbf{c}(\lambda_{i-1}, \lambda_i))$-position.

(1.6) Example.

$$\mathrm{STab}^{(2,l)}([2^1 3^1]/[1]) = \begin{cases} \left\{ \begin{smallmatrix} 12 \\ 34 \end{smallmatrix} \; \begin{smallmatrix} 13 \\ 24 \end{smallmatrix} \; \begin{smallmatrix} 23 \\ 14 \end{smallmatrix} \; \begin{smallmatrix} 14 \\ 23 \end{smallmatrix} \; \begin{smallmatrix} 24 \\ 13 \end{smallmatrix} \right\} & \text{if } l \geq 5, \\ \left\{ \begin{smallmatrix} 13 \\ 24 \end{smallmatrix} \; \begin{smallmatrix} 23 \\ 14 \end{smallmatrix} \; \begin{smallmatrix} 14 \\ 23 \end{smallmatrix} \; \begin{smallmatrix} 24 \\ 13 \end{smallmatrix} \right\} & \text{if } l = 4, \\ \left\{ \begin{smallmatrix} 24 \\ 13 \end{smallmatrix} \right\} & \text{if } l = 3 \end{cases}$$

(1.7) For $\mu \subset^{(k,l)} \lambda$, and for i, $0 \leq i \leq \mathbf{n}(\lambda/\mu)$, let $h_i : \mathrm{STab}^{(k,l)}(\lambda/\mu) \to \Lambda_{\mathbf{n}(\mu)+i}^{(k,l)}$ be the map such that $h_i(x)$ is the i-th component of x, and let $f_i : \mathrm{STab}^{(k,l)}(\lambda/\mu) \to \mathrm{STab}^{(k,l)}(\lambda/\mu)$ be the map such that $f_i(\mathbf{t}) = (h_0(\mathbf{t}), \ldots, h_{i-1}(\mathbf{t}), \nu, h_{i+1}(\mathbf{t}), \ldots, h_{\mathbf{n}(\lambda/\mu)})$ if there exists an element ν of $\Lambda^{(k,l)}$ with $h_{i-1}(\mathbf{t}) \subset_{+1}^{(k,l)} \nu \subset_{+1}^{(k,l)} h_{i+1}(\mathbf{t})$ and $\nu \neq h_i(\mathbf{t})$, and $f_i(\mathbf{t}) = \mathbf{t}$ otherwise. For $\alpha \subset_{+1}^{(k,l)} \beta \subset_{+1}^{(k,l)} \gamma$, put

$$d(\alpha, \beta, \gamma) = \mathbf{c}(\alpha, \beta) - \mathbf{c}(\beta, \gamma) + \mathbf{r}(\beta, \gamma) - \mathbf{r}(\alpha, \beta).$$

For $\mathbf{t} \in \mathrm{STab}^{(k,l)}(\lambda/\mu)$ and i, $1 \leq i \leq \mathbf{n}(\lambda/\mu) - 1$, put $\mathbf{d}(\mathbf{t}; i) = d(h_{i-1}(\mathbf{t}), h_i(\mathbf{t}), h_{i+1}(\mathbf{t}))$.

(1.8) For $q \in \mathbf{C}^{\times}$, and for $d \in \mathbf{Z}$, $1 \leq |d| \leq \mathbf{l}(q) - 1$, put

$$b_d(q) = -\lim_{z \to q} \frac{1-z}{1-z^d},$$

and let $c_d(q) \in \{w \in \mathbf{C} | \mathrm{Im}\, w > 0\} \cup \{x \in \mathbf{R} | x \geq 0\}$ be such that

$$c_d(q)^2 = \lim_{z \to q} \frac{z(1-z^{d-1})(1-z^{d+1})}{(1-z^d)^2}.$$

Note that $b_1(q) = -1$, $b_{-1}(q) = q$.

Theorem (1.9) ([Wenzl]). (i) *Let $q \in \mathbf{C}^\times$, and let $k \in \mathbf{N}$ be such that $k \le \mathbf{l}(q) - 1$. Let μ, $\lambda \in \Lambda^{(k,\mathbf{l}(q))}$ be such that $\mu \subset^{(k,\mathbf{l}(q))} \lambda$. Let $V_{\lambda/\mu}^{(k,\mathbf{l}(q))}$ be a $\mathbf{C}$-vector space with a basis $\{ v_{\mathbf{t}} \,|\, \mathbf{t} \in \mathrm{STab}^{(k,\mathbf{l}(q))}(\lambda/\mu)\}$. Then there exists a representation $\pi_{\lambda/\mu}^{(k,\mathbf{l}(q))} : \mathbf{H}_{\mathbf{n}(\lambda/\mu)}(q) \to$ $\mathbf{End}(V_{\lambda/\mu}^{(k,\mathbf{l}(q))})$ of $\mathbf{H}_{\mathbf{n}(\lambda/\mu)}(q)$ such that*

$$\pi_{\lambda/\mu}^{(k,\mathbf{l}(q))}(T_i)v_{\mathbf{t}} = \begin{cases} b_{\mathbf{d}(\mathbf{t};i)}(q)v_{\mathbf{t}} + c_{\mathbf{d}(\mathbf{t};i)}(q)v_{f_i(\mathbf{t})} & \text{if } f_i(\mathbf{t}) \neq \mathbf{t}, \\ b_{\mathbf{d}(\mathbf{t};i)}(q)v_{\mathbf{t}} & \text{otherwise} \end{cases}$$

$(1 \le i \le \mathbf{n}(\lambda/\mu) - 1)$.

(ii) *For $\lambda \in \Lambda^{(\mathbf{l}(q))}$, let $\pi_\lambda^{(\mathbf{l}(q))} = \pi_{\lambda/\phi}^{(\mathbf{k}(\lambda),\mathbf{l}(q))}$. Then $\pi_\lambda^{(\mathbf{l}(q))}$ is irreducible. For μ, $\lambda \in \Lambda_n^{(\mathbf{l}(q))}$, $\mu \neq \lambda$, $\pi_\mu^{(\mathbf{l}(q))}$ and $\pi_\lambda^{(\mathbf{l}(q))}$ are not equivalent.*

(iii) *Let $\mathbf{l}(q) > n$. Then $\mathbf{H}_n(q)$ is semisimple, $\Lambda_n^{(\mathbf{l}(q))} = \Lambda_n$, and $\{ \pi_\lambda^{(\mathbf{l}(q))} \,|\, \lambda \in \Lambda_n\}$ is a complete set of irreducible representations of $\mathbf{H}_n(q)$.*

§2. Note on irreducible characters

(2.1) Denote by $\mathbf{H}_n(q)^*$ the $\mathbf{C}$-vector space of $\mathbf{C}$-linear maps of $\mathbf{H}_n(q)$ into $\mathbf{C}$. If an element $f \in \mathbf{H}_n(q)^*$ satisfies the condition such that

$$\forall X, \forall Y \in \mathbf{H}_n(q)\ f(XY - YX) = 0,$$

we call f a *class function* of $\mathbf{H}_n(q)$. Denote by $\mathbf{CF}(\mathbf{H}_n(q))$ the set of the class functions of $\mathbf{H}_n(q)$. For $\lambda \in \Lambda_n^{(\mathbf{l}(q))}$, denote by $\chi_\lambda^{(\mathbf{l}(q))}$ the character $\mathrm{Trace} \circ \pi_\lambda^{(\mathbf{l}(q))}$. It is clear that $\chi_\lambda^{(\mathbf{l}(q))} \in \mathbf{CF}(\mathbf{H}_n(q))$.

For k, $m \in \mathbf{N}$, $1 \le m < k \le n$, put

$$\begin{aligned} S_{m,k} &= (T_{k-m} \cdots T_1)(T_{k-m+1} \cdots T_2) \cdots (T_{k-1} \cdots T_m) \\ &= (T_{k-m} \cdots T_{k-1})(T_{k-m-1} \cdots T_{k-2}) \cdots (T_1 \cdots T_m). \end{aligned}$$

Then we have $S_{m,k}T_j = T_{j-m}S_{m,k}$ $(m+1 \le j \le k-1)$ and $S_{m,k}T_i = T_{k-m+i}S_{m,k}$ $(1 \le i \le m-1)$.

For a, $n \in \mathbf{N}$, $1 < a < n$, denote by $\iota_{a,n}$ the monomorphism of $\mathbf{H}_a(q)$ into $\mathbf{H}_n(q)$ such that $\iota_{a,n}(T_i) = T_i$ $(1 \le i \le a-1)$. For $X \in \mathbf{H}_a(q)$, $\iota_{a,n}(X)$ will also be denoted by X. For k, $m \in \mathbf{N}$, $1 \le m < k \le n$, and for $X \in \mathbf{H}_a(q)$, $Y \in \mathbf{H}_{k-m}(q)$, put

$$X \sharp Y = X S_{k-m,k} Y S_{k-m,k}^{-1} \in \mathbf{H}_n(q).$$

For $f \in \mathbf{CF}(\mathbf{H}_n(q))$, we have

$$f(X \sharp Y) = f(Y S_{k-m,k}^{-1} X S_{k-m,k}) = f(Y S_{m,k} X S_{m,k}^{-1}) = f(Y \sharp X).$$

We also have $f((X \sharp Y) \sharp Z) = f(X \sharp (Y \sharp Z))$.

Let $\chi^{(\mathbf{k}(\lambda),\mathbf{l}(q))}_{\lambda/\mu} = \mathrm{Trace} \circ \pi^{(\mathbf{k}(\lambda),\mathbf{l}(q))}_{\lambda/\mu}$. The following lemma is easily obtained by the definition of $\pi^{(l)}$.

Lemma (2.2) Let k, $m \in \mathbf{N}$, $1 < m < n$, and $X \in \mathbf{H}_m(q)$, $Y \in \mathbf{H}_{k-n}(q)$. Let $\lambda \in \Lambda^{(\mathbf{l}(q))}$. Then

$$\pi^{(\mathbf{l}(q))}_\lambda(X\sharp Y) \cong \bigoplus_{\mu\in\Lambda_m^{(\mathbf{k}(\lambda),\mathbf{l}(q))}} \pi^{(\mathbf{l}(q))}_\mu(X) \otimes \pi^{(\mathbf{k}(\lambda),\mathbf{l}(q))}_{\lambda/\mu}(Y).$$

In particular,

$$\chi^{(\mathbf{l}(q))}_\lambda(X\sharp Y) = \sum_{\mu\in\Lambda_m^{(\mathbf{k}(\lambda),\mathbf{l}(q))}} \chi^{(\mathbf{l}(q))}_\mu(X)\chi^{(\mathbf{k}(\lambda),\mathbf{l}(q))}_{\lambda/\mu}(Y).$$

(2.3) For $\lambda \in \Lambda^{(\mathbf{l}(q))}_n$, denote by $e^{(\mathbf{l}(q))}_\lambda$ a primitive idempotent of $\mathbf{H}_n(q)$ corresponding to the irreducible representation $\pi^{(\mathbf{l}(q))}_\lambda$. For k, m, $n \in \mathbf{N}$, $1 < m < n$, and for $\mu \in \Lambda^{(k,\mathbf{l}(q))}_m$, $\nu \in \Lambda^{(k,\mathbf{l}(q))}_{n-m}$, $\lambda \in \Lambda^{(k,\mathbf{l}(q))}_n$, μ, $\nu \subset^{(k,\mathbf{l}(q))} \lambda$, define the integer $d^{(k,\mathbf{l}(q))}(\lambda;\mu,\nu)$ to be $\chi^{(k,\mathbf{l}(q))}_\lambda(e^{(\mathbf{l}(q))}_\mu \sharp e^{(\mathbf{l}(q))}_\nu) = \chi^{(k,\mathbf{l}(q))}_{\lambda/\mu}(e^{(\mathbf{l}(q))}_\nu)$. Denote by $\mathbf{S}(\mathbf{N})$ the group of the bijective maps of $\mathbf{N}$ onto itsself. Let $s_i \in \mathbf{S}(\mathbf{N})$ $(i \in \mathbf{N})$ be the map such that $p_i(s_i(\alpha)) = p_{i+1}(\alpha)$, $p_{i+1}(s_i(\alpha)) = p_i(\alpha)$ and $p_j(s_i(\alpha)) = p_j(\alpha)$ $(j \neq i,\ i+1)$. Let $s_0^{(k,l)} \in \mathbf{S}(\mathbf{N})$ $(i \in \mathbf{N})$ be the map such that $p_1(s_0^{(k,l)}(\alpha)) = p_k(\alpha) + l$, $p_k(s_0^{(k,l)}(\alpha)) = p_1(\alpha) - l$ and $p_j(s_0^{(k,l)}(\alpha)) = p_j(\alpha)$ $(j \neq 1,\ k)$. Let $W^{(k,l)}$ be the subgroup of $\mathbf{S}(\mathbf{N})$ generated by $s_0^{(k,l)}$ and s_i $(1 \le i \le k)$. Define $\delta(k) \in \lambda$ to be $(k-1, k-2, \ldots, 1, 0, \ldots)$. Goodman-Wenzl [GW] proved:

Theorem (2.4)([GW])

$$d^{(k,l)}(\lambda;\mu,\nu) = \sum_{w\in W^{(k,l)}} \mathrm{sgn}(w) d^{(k,l)}(w(\lambda+\delta(k)) - \delta(k);\mu,\nu).$$

Since this identity is coincides with the Kac-Walton one, $d^{(k,l)}(\lambda;\mu,\nu)$ is the same as the SU(k)-fusion coefficient with level $l-k$ (see [GW]).

(2.5) An element α of $\mathbf{Z}_+^\infty$ is called a *composition* if there exists $j \in \mathbf{N}$ such that $p_i(\alpha) > 0$ for $i < j$ and $p_k(\alpha) = 0$ for $k \ge j$. Denote by Ω the set of the compositions. Let $\omega : \Omega \to \Lambda$ be the map such that $\omega(\alpha) = \sigma(\alpha)$ for some $\sigma \in \mathbf{S}(\mathbf{N})$. Let $\Omega_n = \omega^{-1}(\Lambda_n)$ and $\Omega' = \omega^{-1}(\Lambda')$.

(2.6) The maps $\mathbf{k} \circ \omega$, $\mathbf{n} \circ \omega$ of Ω into $\mathbf{Z}_+$ will also be denoted by $\mathbf{k}$, $\mathbf{n}$ respectively. For $\alpha \in \Omega'$ and i, $1 \le i \le \mathbf{k}(\alpha)$, let $x_i(\alpha) = \sum_{1\le j<i} p_j(\alpha) \in \mathbf{Z}_+$, and let $X(\alpha;i)$ be the element of $\mathbf{H}_{\mathbf{n}(\alpha)}(q)$ such that $X(\alpha;i) = T_1T_2\cdots T_{p_i(\alpha)-1}$ if $p_i(\alpha) \ge 2$, and $X(\alpha;i) = 1$ if $p_i(\alpha) = 1$. Put

$$X(\alpha) = X(\alpha;1)\sharp\cdots\sharp X(\alpha;\mathbf{k}(\alpha)).$$

By the following theorem proved by Ram[Ram], we see that any class function $f : \mathbf{H}_n(q) \to \mathbf{C}$ is determined only by the values $f(X(\lambda))$ $(\lambda \in \Lambda_n)$.

Theorem (2.7)([Ram]). *Let $n \in \mathbf{N}$. Then*

$$^{\forall}X \in \mathbf{H}_n(q)\ ^{\exists}x_\lambda \in \mathbf{C}\ (\lambda \in \Lambda_n)\ ^{\forall}f \in \mathbf{CF}(\mathbf{H}_n(q)) \quad f(X) = \sum_{\lambda \in \Lambda_n} x_\lambda f(C(\lambda))$$

We can calculate the coefficients x_λ's via an inductive process.

(2.8) For μ, $\lambda \in \Lambda^{(k,\mathbf{l}(q))}$, $\mu \subset^{(k,\mathbf{l}(q))} \lambda$, put $\Delta^{(k,\mathbf{l}(q))}(\lambda/\mu) = \chi^{(k,\mathbf{l}(q))}(\lambda/\mu)(X([\mathbf{n}(\lambda/\mu)^1]))$. It turns out that

$$\Delta^{(k,\mathbf{l}(q))}(\lambda/\mu) = \sum_{\mathbf{t} \in \mathbf{STab}^{(k,\mathbf{l}(q))}(\lambda/\mu)} \prod_{1 \leq i \leq \mathbf{n}(\lambda/\mu)} b_{\mathbf{d}(\mathbf{t};i)}(q).$$

As an immediate consequence of Lemma (2.2), we have:

Theorem (2.9) *Let $\lambda \in \Lambda_n^{(k,l)}$. Let $\alpha \in \Omega_n$. Then*

$$\chi_\lambda^{(k,l)}(X(\alpha)) = \sum_{\substack{\phi = \mu_0 \subset^{(k,\mathbf{l}(q))} \cdots \subset^{(k,\mathbf{l}(q))} \mu_r = \lambda \\ p_i(\alpha) = \mathbf{n}(\mu_i/\mu_{i-1})}} \prod_{i=1}^{r} \Delta^{(k,\mathbf{l}(q))}(\mu_i/\mu_{i-1}).$$

Ram[Ram] gave an explicit formula of $\Delta^{(k,\mathbf{l}(q))}(\mu_i/\mu_{i-1})$ for $\mathbf{l}(q) > \mathbf{n}(\lambda)$.

§3. Main result

(3.1) Let $l = \mathbf{l}(q)$. In §3, $a \equiv b$ means $a - b \in l\mathbf{Z}$. For $\lambda \in \Lambda^{(k,\mathbf{l}(q))}$, $\Delta(\lambda/\phi)$ will also be denoted by $\{p_1(\lambda), \ldots, p_{\mathbf{k}(\lambda)}(\lambda)\}$. If $\mathbf{k}(\lambda) = 2$, we have:

$$\{r, r-a\} = \begin{cases} -q^{-a-3} & r \equiv -1 \\ q^{-a-2} & a \geq 1,\ r \equiv 0 \\ q^{-a-2} + q^{a-1} & a = 0,\ r \equiv a \\ -q^{a} & r \equiv a+1 \\ 0 & \text{otherwise} \end{cases}$$

(3.2) If $\mathbf{k}(\lambda) = 3$, we have:

$\{r,r,r\}$	$l \geq 5$	$q^{-4}\,(r\equiv -1)$, $q^{-3}+q^{-2}+q^{-1}\,(r\equiv 0)$, $1\,(r\equiv 1)$, $0\,$(oth.)
—	$l=4$	$1\,(r\equiv 1,3)$, $-1\,(r\equiv 0,2)$
$\{r,r,r-1\}$	$l \geq 6$	$q^{-5}\,(r\equiv -1)$, $q^{-4}\,(r\equiv 0)$, $-1\,(r\equiv 1)$, $0\,$(oth.)
—	$l=5$	$1\,(r\equiv -1)$, $q\,(r\equiv 0)$, $-1\,(r\equiv 1)$, $0\,(r\equiv 2)$, $-q\,(r\equiv -2)$
—	$l=4$	$1\,(r\equiv 0,2)$, $-1\,(r\equiv 1,3)$
$\{r,r,r-a\}$ $(a\geq 2)$	$l\geq a+5$	$q^{-(a+4)}\,(r\equiv -1)$, $q^{-(a+3)}\,(r\equiv 0)$, $0\,$(oth.)
—	$l=a+4$	$q\,(r\equiv 0)$, $-q^{a}\,(r\equiv -2)$, $1\,(r\equiv -1)$, $0\,$(oth.)
—	$l=a+3$	$1\,(r\equiv 0)$, $q^{a-1}\,(r\equiv -2)$, $q^{a}+q^{a+1}+q^{a+2}\,(r\equiv -1)$, $0\,$(oth.)
$\{r,r-1,r-1\}$	$l\geq 6$	$-q^{-4}\,(r\equiv 0)$, $1\,(r\equiv 1)$, $q\,(r\equiv 2)$, $0\,$(oth.)
—	$l=5$	$-q\,(r\equiv 0)$, $1\,(r\equiv 1)$, $q\,(r\equiv 2)$, $-1\,(r\equiv 3)$, $0\,(r\equiv 4)$
—	$l=4$	$1\,(r\equiv 1,3)$, $-1\,(r\equiv 0,2)$
$\{r,r-a,r-a\}$ $(a\geq 2)$	$l\geq a+5$	$q^{a-1}\,(r\equiv a)$, $q^{a}\,(r\equiv a+1)$, $0\,$(oth.)
—	$l=a+4$	$q^{a-1}\,(r\equiv a)$, $q^{a}\,(r\equiv a+1)$, $-1\,(r\equiv a+2)$, $0\,$(oth.)
—	$l=a+3$	$q^{a-1}\,(r\equiv a)$, $q^{a}+q^{a+1}+q^{a+2}\,(r\equiv a+1)$, $1\,(r\equiv a+2)$, $0\,$(oth.)
$\{r,r-1,r-2\}$	$l\geq 7$	$-q^{-5}\,(r\equiv 0)$, $-q\,(r\equiv 2)$, $0\,$(oth.)
—	$l=6$	$-q\,(r\equiv 0,2,4)$, $0\,$(oth.)
—	$l=5$	$-1\,(r\equiv 0)$, $0\,(r\equiv 1)$, $-q\,(r\equiv 2)$, $1\,(r\equiv -2)$, $q\,(r\equiv -1)$
$\{r,r-1,r-a\}$ $(a\geq 3)$	$l\geq a+5$	$-q^{-(a+3)}\,(r\equiv 0)$, $0\,$(oth.)
—	$l=a+4$	$-q\,(r\equiv 0)$, $-q^{a-1}\,(r\equiv -2)$, $0\,$(oth.)
—	$l=a+3$	$-1\,(r\equiv 0)$, $q^{a-2}\,(r\equiv -2)$, $q^{a-1}\,(r\equiv -1)$, $0\,$(oth.)
$\{r,r-a+1,r-a\}$ $(a\geq 3)$	$l\geq a+5$	$-q^{a-1}\,(r\equiv a)$, $0\,$(oth.)
—	$l=a+4$	$-q^{a-1}\,(r\equiv a)$, $-q\,(r\equiv a+2)$, $0\,$(oth.)
—	$l=a+3$	$-q^{a-1}\,(r\equiv a)$, $1\,(r\equiv a+1)$, $q\,(r\equiv a+2)$, $0\,$(oth.)
$\{r,r-b,r-a\}$ $(a-2\geq b\geq 2)$	$l\geq a+5$	0
—	$l=a+4$	$-q^{a-b}\,(r\equiv -2)$, $0\,$(oth.)
—	$l=a+3$	$q^{a-b-1}\,(r\equiv -2)$, $q^{a-b}\,(r\equiv -1)$, $0\,$(oth.)

§4. On Littlewood-Richardson rule

(4.1) In this section, we use terminology in [Macdonald]. Let $\mu\subset\lambda$. Denote by $\mathbf{LP}(\lambda/\mu)$ the set of the lattice permutations of shape λ/μ. For $\mathbf{z}\in\mathbf{LP}(\lambda/\mu)$ and $\nu\in\Lambda_{\mathbf{n}(\lambda/\mu)}$, let $\mathbf{LP}(\lambda/\mu;\nu)$ be the set of the lattice permutations of shape λ/μ and weight ν. It is well-known that the cardinarity of $\mathbf{LP}(\lambda/\mu;\nu)$ is equal to $d^{(k,\mathbf{l}(q))}(\lambda;\mu,\nu)$ for $\mathbf{l}(q)>\mathbf{n}(\lambda)$. See [Macdonald].

(4.2) The set $\mathbf{LP}([2^1 3^1 4^1]/[2])$ consists of the lattice permutations

$$\begin{matrix} & & 1 & 1 \\ 1 & 1 & 2 & \\ 2 & 2 & & \end{matrix}\;,\quad \begin{matrix} & & 1 & 1 \\ 1 & 1 & 2 & \\ 2 & 3 & & \end{matrix}\;,\quad \begin{matrix} & & 1 & 1 \\ 1 & 2 & 2 & \\ 2 & 3 & & \end{matrix}\;,\quad \begin{matrix} & & 1 & 1 \\ 1 & 2 & 2 & \\ 3 & 3 & & \end{matrix}\;.$$

The weights of the first, the second, the third, and the fourth lattice permutations are $[3^1 4^1]$, $[1^1 2^1 4^1]$, $[1^1 3^2]$, $[2^2 3^1]$, respectively.

Theorem (4.3) *Let $\mu,\ \lambda \in \Lambda^{(3,l)}$ be such that $\mu \subset^{(3,l)} \lambda$. Assume $p_1(\mu) - p_2(\mu) \leq 1$. For $\mathbf{z} \in \mathbf{LP}(\lambda/\mu)$, let $\mathbf{z}_{ij}$ be the Arabic figure at (i,j)-position, and let $\mathbf{bot}(\lambda/\mu; a)$ be the number of j's such that $\mathbf{z}_{3j} = a$. Denote by $\mathbf{Y}^{(3,\mathbf{l}(q))}(\lambda/\mu)$ the set of the lattice permutations $\mathbf{z} \in \mathbf{LP}(\lambda/\mu)$ such that*

(1) $\mathbf{bot}(\lambda/\mu; 1) + 1 + p_1(\lambda) - p_3(\lambda) \leq \mathbf{l}(q) - 3$ *if* $p_1(\mu) = p_2(\mu) + 1$, $\mathbf{z}_{2,p_1(\mu)} = 1$, $\mathbf{z}_{3,p_1(\mu)} = 2$,

(2) $\mathbf{bot}(\lambda/\mu; 1) + 1 + p_1(\lambda) - p_3(\lambda) \leq \mathbf{l}(q) - 3$ *if* $p_1(\mu) = p_2(\mu) + 1$, $\mathbf{z}_{2,p_1(\mu)} = 1$, $\mathbf{z}_{3,p_1(\mu)} \neq 2$, $\mathbf{bot}(\lambda/\mu; 2) = p_1(\lambda) - p_2(\lambda) + 1$,

(3) $\mathbf{bot}(\lambda/\mu; 1) + 1 + p_1(\lambda) - p_3(\lambda) \leq \mathbf{l}(q) - 3$ *otherwise.*

Then $d^{(3,\mathbf{l}(q))}(\lambda; \mu, \nu)$ is equal to the number of the elements of $\mathbf{Y}^{(3,\mathbf{l}(q))}(\lambda/\mu)$ whose weights are ν.

Example (10.3). *On* $Y^{(3,\mathbf{l}(q))}([7^1 10^1 12^1]/[5^2])$. We shall only write $a\,b\,c\,d\,e$ for

$$\mathbf{z} = \begin{array}{ccccccccccccc} & & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & & & 2 & 2 & 2 & 2 & 2 & & \\ a & b & c & d & e & 3 & 3 & & & & & \end{array}.$$

(x, y, z) denotes weight. Then it consists of:

$\mathbf{l}(q) \geq 8$, $(7,7,5)$ 22333,

$\mathbf{l}(q) \geq 9$, $(8,6,5)$ 12333, $(8,7,4)$ 12233,

$\mathbf{l}(q) \geq 10$, $(9,5,5)$ 11333, $(9,6,4)$ 11233, $(9,7,3)$ 11223,

$\mathbf{l}(q) \geq 11$, $(10,5,4)$ 11133, $(10,6,3)$ 11123, $(10,7,2)$ 11122,

$\mathbf{l}(q) \geq 12$, $(11,5,3)$ 11113, $(11,6,2)$ 11112,

$\mathbf{l}(q) \geq 13$, $(12,5,2)$ 11111,

Example (10.4). *On* $Y^{(3,\mathbf{l}(q))}([6^1 9^1 11^1]/[3^1 4^1])$.

$$\mathbf{z} = \begin{array}{ccccccccccc} & & & & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ & & & a & 2 & 2 & 2 & 2 & 2 & & \\ b & c & d & e & 3 & 3 & & & & & \end{array}$$

$\mathbf{l}(q) \geq 8,\ (7,6,6)^{2}_{3333},\ (7,7,5)^{2}_{2333},\ (8,6,5)^{1}_{2333},\ (8,7,4)^{1}_{2233},$

$\mathbf{l}(q) \geq 9,\ (8,6,5)^{2}_{1333},\ (8,7,4)^{2}_{1233},\ (8,8,3)^{1}_{2223},\ (9,5,5)^{1}_{1333},\ (9,6,4)^{1}_{1233},\ (9,7,3)^{1}_{1223},$

$\mathbf{l}(q) \geq 10,\ (9,6,4)^{2}_{1133},\ (9,7,3)^{2}_{1123},\ (9,8,2)^{1}_{1222},\ (10,5,4)^{1}_{1133},\ (10,6,3)^{1}_{1123},$

$\mathbf{l}(q) \geq 11,\ (10,6,3)^{2}_{1113},\ (10,7,2)^{1}_{1122},\ (11,5,3)^{1}_{1113},$

$\mathbf{l}(q) \geq 12,\ (11,6,2)^{1}_{1112}$

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[河野聖司] 河野聖司, 平成１０年度大阪大学理学研究科修士論文 (1999)

Hiroyuki Yamane
Osaka university,
Dept. of Math.,
Toyonaka Osaka 560
Japan
yamane@math.sci.osaka-u.ac.jp